CLASSICAL YANG- BAXTER EQUATION AND LOW DIMENSIONAL TRIANGULAR LIE BIALGEBRAS OVER ARBITRARY FIELD *

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Abstract

Let L be a Lie algebra over arbitrary field k with dim L =3 and dim L' =2. All solutions of constant classical Yang- Baxter equation (CYBE) in Lie algebra L are obtained and the necessary conditions which $(L, [\ ], \Delta_r, r)$ is a coboundary (or triangular) Lie bialgebra are given.

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1 Introduction

The concept and structures of Lie coalgebras were introduced and studied by W. Michaelis in [6, 7]. V.G.Drinfel'd and A. A. Belavin in [1, 2] introduced the notion of triangular, coboundary L associated to a solution $r \in L \otimes L$ of the CYBE and gave a classification of solutions of CYBE with parameter for simple Lie algebras. W.Michaelis in [5] obtained the structure of a triangular, coboundary Lie bialgebra on any Lie algebra containing linearly independent elements a and b satisfying $[a, b] = \alpha b$ for some non-zero $\alpha \in k$ by setting $r = a \otimes b - b \otimes a$.

The Yang-Baxter equation first came up in a paper by Yang as factorition condition of the scattering S-matrix in the many-body problem in one dimension and in work of Baxter on exactly solvable models in statistical mechanics. It has been playing an important role in mathematics and physics (see [1, 9]).Attempts to find solutions The Yang- Baxter

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equation in a systematic way have let to the theory of quantum groups. The Yang-Baxter equation is of many forms. The classical Yang- Baxter equation is one.

In many applications one need to know the solutions of classical Yang- Baxter equation and know if a Lie algebra is a coboundary Lie bialgebra or a triangular Lie bialgebras. A systematic study of low dimensional Lie algebras, specially, of those Lie algebras that play a role in physics(as e.g. sl(2, C), or the Heisenberg algebra), is very useful.The author [10, 11] obtained all solutions of constant classical Yang- Baxter equation (CYBE) in Lie algebra L and give the sufficient and necessary conditions which ($L, [\ ], \Delta_r, r$)is a coboundary (or triangular) Lie bialgebra with dim $L \leq 3$ except the below case : L is a Lie algebra over arbitrary field k with dim L =3 and dim $L^{'} = 2$ .We shall resolve the problem in this paper.

All of the notations in this paper are the same as in [10].

If k is not algebraically closed, let P be algebraic closure of k. we can construct a Lie algebra $L_P = P \otimes L$ over P, as in [3, Section 8].

By [3, P11–14], we have that

Lemma 1.1 *Let L be a vector space over k. Then L is a Lie algebra over field k with dim L =3 and dim $L^{'} = 2$ iff there is a basis e_1, e_2, e_3 in L such that $[e_1, e_2]$ =0, $[e_1, e_3] = \alpha e_1 + \beta e_2$, $[e_2, e_3] = \gamma e_1 + \delta e_2$, where $\alpha, \beta, \gamma, \delta \in k$, and $\alpha\delta - \beta\gamma \neq 0$.*

In this paper, we only study the Lie algebra L in Lemma 1.1. Set $A = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}$. Thus A is similar to $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ or $\begin{pmatrix} \lambda_1 & 0 \\ 1 & \lambda_1 \end{pmatrix}$ in the algebraic closure P of k. Therefore, there is an invertible matrix D over P such that $AD = D \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, or $AD = D \begin{pmatrix} \lambda_1 & 0 \\ 1 & \lambda_2 \end{pmatrix}$. Let $Q = \begin{pmatrix} D & 0 \\ 0 & \frac{1}{\lambda_1} \end{pmatrix}$ and $(e_1^{'}, e_2^{'}, e_3^{'}) = (e_1, e_2, e_3)Q$. By computation, we have that $[e_1^{'}, e_2^{'}] = 0$, $[e_1^{'}, e_3^{'}] = e_1^{'} + \beta^{'} e_2^{'}$, $[e_2^{'}, e_3^{'}] = \delta^{'} e_2^{'}$, where $\beta^{'} = 0$ and $\delta^{'} = \frac{\lambda_1}{\lambda_2}$ when A is similar to $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$; $\beta^{'} = \frac{1}{\lambda_1}$ and $\delta^{'} = 1$ when A is similar to $\begin{pmatrix} \lambda_1 & 0 \\ 1 & \lambda_1 \end{pmatrix}$. Let $Q = (q_{ij})_{3\times 3}$ and $Q^{-1} = (\bar{q}_{ij})_{3\times 3}$. If $r = \sum_{i,j=1}^{3} k_{ij}(e_i \otimes e_j) = \sum_{i,j=1}^{3} k_{ij}^{'}(e_i^{'} \otimes e_j^{'})$, where $k_{ij} \in k$, $k_{ij}^{'} \in P$ for i, j =1, 2, 3, then

$$k_{ij}^{'} = \sum_{m,n}^{3} k_{mn}\overline{q}_{im}\overline{q}_{jn} \text{ and } k_{ij} = \sum_{m,n}^{3} k_{mn}^{'} q_{im} q_{jn}$$

for $i, j = 1, 2, 3$. Obviously, $k_{33} = k_{33}^{'}$.

Lemma 1.2 *(i) $k_{i3} = k_{3i}$, for i =1, 2, 3 iff $k_{i3}^{'} = k_{3i}^{'}$ for i = 1, 2, 3;*

(ii) $k_{i3} = -k_{3i}$ *for* i *=1, 2, 3 iff* $k'_{i3} = -k'_{3i}$ *for* i *=1, 2, 3;*
(iii) $k_{ij} = k_{ji}$ *for* i, j *=1, 2, 3 iff* $k'_{ij} = k'_{ji}$ *for* $i, j = 1, 2, 3$*;*
(iv) $k_{ij} = -k_{ji}$ *for* i, j *=1, 2, 3 iff* $k'_{ij} = -k'_{ji}$ *for* i, j *=1, 2, 3.*

Proof (i) If $k'_{i3} = k'_{3i}$ for i =1, 2, 3, we see that

$$\begin{aligned} k_{i3} &= \sum_{m,n}^{3} k'_{mn} q_{im} q_{3n} \\ &= \sum_{m}^{3} k'_{m3} q_{im} q_{33} \quad (\text{ since } q_{31} = q_{32} = 0) \\ &= \sum_{m}^{3} k'_{3m} q_{im} q_{33} \quad (\text{ by assumption}) \\ &= \sum_{m}^{3} k'_{3m} q_{33} q_{im} \\ &= \sum_{m,n}^{3} k'_{mn} q_{3n} q_{im} \\ &= k_{3i}. \end{aligned}$$

Therefore, $k_{i3} = k_{3i}$ for i =1, 2, 3. The others can be proved similarly. □

2 The solutions of CYBE with char $k \neq 2$

In this section, we find the general solution of CYBE for Lie algebra L with dim L =3 and dim $L' = 2$, where char $k \neq 2$.

Theorem 2.1 *Let* L *be a Lie algebra with a basis* e_1, e_2, e_3 *such that* $[e_1, e_2]$ *=0,* $[e_1, e_3] = \alpha e_1 + \beta e_2$*,* $[e_2, e_3] = \gamma e_1 + \delta e_2$*, where* $\alpha, \beta, \gamma, \delta \in k$*, and* $\alpha\delta - \beta\gamma \neq 0$*. Let* $p, q, s, t, u, v, x, y, z \in k$*. Then* r *is a solution of CYBE iff* r *is strongly symmetric, or* $r = p(e_1 \otimes e_2) + q(e_2 \otimes e_1) + s(e_1 \otimes e_3) - s(e_3 \otimes e_1) + u(e_2 \otimes e_3) - u(e_3 \otimes e_2) + x(e_1 \otimes e_1) + y(e_2 \otimes e_2)$ *with* $s(2\alpha x + \gamma(p+q)) = u(2\delta y + \beta(q+p)) = u(2\alpha x + \gamma(q+p)) = s(2\delta y + \beta(q+p)) = (\alpha - \delta)us + \gamma u^2 - \beta s^2 = s(2\gamma y + 2\beta x + (\alpha+\delta)(q+p)) = u(2\gamma y + 2\beta x + (\alpha+\delta)(p+q)) = 0$.

Proof Let $r = \sum_{i,j=1}^{3} k_{ij}(e_i \otimes e_j) \in L \otimes L$, and $k_{ij} \in k$, with i, j =1, 2, 3. By computation, for all i, j, n =1, 2, 3, we have that the cofficient of $e_j \otimes e_i \otimes e_i$ in $[r^{12}, r^{23}]$ is zero and the cofficient of $e_i \otimes e_i \otimes e_j$ in $[r^{13}, r^{23}]$ is zero.

We can obtain the following equations by seeing the cofficient of $e_i \otimes e_j \otimes e_n$ in $[r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}]$, as in [10, Proposition 2.6]. To simplify notation, let $k_{11} = x, k_{22} = y, k_{33} = z, k_{12} = p, k_{21} = q, k_{13} = s, k_{31} = t, k_{23} = u, k_{32} = v$.

(1) $-\alpha sx + \alpha xt - \gamma sq + \gamma pt = 0$;
(2) $-\beta up + \beta qv - \delta uy + \delta yv = 0$;

(3) $-\alpha vs+\alpha pz-\gamma uv+\gamma yz-\beta s^2+\beta xz-\delta su+\delta zp=0$;

(4) $-\beta xz+\beta st-\delta zq+\delta ut-\alpha ut+\alpha qz-\gamma uv+\gamma yz=0$;

(5) $-\alpha zp+\alpha tv-\gamma zy+\gamma uv-\beta zx+\beta ts-\delta zp+\delta vs=0$;

(6) $-\alpha zp+\alpha sv-\gamma zy+\gamma uv-\beta st+\beta xz-\delta sv+\delta zp=0$;

(7) $-\beta zx+\beta tt-\delta zq+\delta vt-\alpha zq+\alpha tu-\gamma zy+\gamma uv=0$;

(8) $-\beta st+\beta xz-\delta tu+\delta qz-\alpha us+\alpha qz-\gamma uu+\gamma yz=0$;

(9) $-\alpha tp+\alpha xv-\gamma ty+\gamma qv-\alpha sp+\alpha vx-\gamma sy+\gamma pv=0$;

(10) $-\alpha ux+\alpha qt-\gamma uq+\gamma yt-\alpha ux+\alpha qs-\gamma up+\gamma ys=0$;

(11) $-\alpha st+\alpha xz-\gamma tu+\gamma qz-\alpha s^2+\alpha xz-\gamma su+ypz=0$;

(12) $-\alpha zx+\alpha tt-\gamma zq+\gamma vt-\alpha zx+\alpha st-\gamma zp+\gamma vs=0$;

(13) $-\beta vx+\beta pt-\delta vq+\delta yt-\beta ux+\beta qt-\delta uq+\delta yt=0$;

(14) $-\beta sp-\beta xv-\delta sy+\delta pv-\beta sq+\beta xu-\delta sy+\delta pu=0$;

(15) $-\beta vs+\beta pz-\delta vu+\delta yz-\beta su+\beta zq-\delta u^2+\delta yz=0$;

(16) $-\beta zp+\beta tv-\delta zy+\delta vv-\beta zq+\beta tu-\delta zy+\delta vu=0$;

(17) $-\gamma zq+\gamma ut-\gamma sv+\gamma pz=0$;

(18) $-\alpha vx+\alpha pt-\gamma vq+\gamma yt-\beta sx+\beta xt-\delta sq+\delta pt-\alpha sq+\alpha xu-\gamma sy+\gamma pu=0$;

(19) $-\beta pt+\beta vx-\delta ty+\delta qv-\alpha up+\alpha qv-\gamma uy+\gamma yv-\beta ux+\beta qs-\delta up+\delta ys=0$;

(20) $-\beta zp+\beta sv-\delta zy+\delta uv-\beta ut+\beta qz-\delta uv+\delta yz=0$;

(21) $-\beta zs+\beta tz-\delta zu+\delta vz=0$;

(22) $-\alpha zs+\alpha tz-\gamma zu+\gamma vz=0$;

It is clear that r is a solution of CYBE iff (1)-(22) hold.

By simple computation, we have the sufficiency. Now we show the necessity, If $k_{33}\neq 0$, then $k_{33}^{'}\neq 0'$ and so r is a strongly symmetric element in $L_P\otimes L_P$ by [10, The proof of Proposition 1.6]. Thus r is a strongly symmetric element in $L\otimes L$. If k_{33} =0, then $k_{33}^{'}$ =0. By [10, Proposition 1.6], we have that $k_{i3}^{'}=-k_{3i}^{'}$ for i =1, 2, 3, which implies that $k_{i3}=-k_{3i}$ for i =1, 2, 3 by Lemma 1.2.

It immediately follows from (1)-(22) that

(23) $s(-2\alpha x-\gamma(q+p))=0$;

(24) $u(2\delta y+\beta(q+p))=0$;

(25) $\gamma u^2-\beta s^2+(\alpha-\delta)us=0$;

(26) $u(2\alpha x+\gamma(q+p))=0$;

(27) $s(2\delta y+\beta(q+p))=0$;

(28) $2\alpha ux-2\gamma ys-2\beta xs+(-s(\alpha+\delta)+\gamma u)(q+p)=0$;

(29) $-2\beta ux+2\delta ys-2\gamma uy+(-u(\alpha+\delta)+\beta s)(q+p)=0$;

By (26), (27), (28) and (29), we have that

(30) $s(2\gamma y+2\beta x+(\alpha+\delta)(q+p))=0$;

(31) $u(2\gamma y+2\beta x+(\alpha+\delta)(p+q))=0$. $\square$

Example 2.2 *Let L be a Lie algebra over real field R with dim L =3 and dim $L^{'} = 2$. If there is complex characteristic root $\lambda_1 = a + bi$ of A and the root is not real, then $r \in L \otimes L$ is a solution of CYBE iff r is strongly symmetric, or $r = p(e_1 \otimes e_2) + q(e_2 \otimes e_1) + x(e_1 \otimes e_1) + y(e_2 \otimes e_2)$ for any $p, q, x, y \in R$.*

Proof. There are two different characteristic roots: $\lambda_1 = a+bi$ and $\lambda_2 = a-bi$, where $a, b \in R$, Thus A must be similar to $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. By Theorem 2.1, we can complete the proof. □

3 The solutions of CYBE with char k =2

In this section, we find the general solution of CYBE for Lie algebra L with dim L =3 and dim $L^{'} = 2$, where char k =2.

Theorem 3.1 *Let L be a Lie algebra with a basis e_1, e_2, e_3 such that $[e_1, e_2]$ =0, $[e_1, e_3] = \alpha e_1 + \beta e_2$, $[e_2, e_3] = \gamma e_1 + \delta e_2$, where $\alpha, \beta, \gamma, \delta \in k$, and $\alpha\delta - \beta\gamma \neq 0$. Let $p, q, s, t, u, v, x, y, z \in k$ and $A = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}$.*

(I) If two characteristic roots of A are equal and A is similar to a diagonal matrix in the algebraic closure P of k, then r is a solution of CYBE in L for any $r \in L \otimes L$;

(II) If the condition in Part (1)does not hold, then r is a solution of CYBE in L iff $r = p(e_1 \otimes e_1) + p(e_2 \otimes e_1) + s(e_1 \otimes e_3) + s(e_3 \otimes e_1) + u(e_2 \otimes e_3) - +u(e_3 \otimes e_2) + x(e_1 \otimes e_1) + y(e_2 \otimes e_2) + z(e_3 \otimes e_3)$ with $\alpha us + \alpha pz + \gamma u^2 + \gamma yz + \beta s^2 + \beta xz + \delta su + \delta zp = 0$ and $z \neq 0$; or $r = p(e_1 \otimes e_2) + q(e_2 \otimes e_1) + s(e_1 \otimes e_3) + s(e_3 \otimes e_1) + u(e_2 \otimes e_3) + u(e_3 \otimes e_2) + x(e_1 \otimes e_1) + y(e_2 \otimes e_2)$ with $s\gamma(p + q) = u\beta(p + q) = u\gamma(p + q) = s\beta(p + q) = (\alpha + \delta)us + \gamma u^2 + \beta s^2 = s(\alpha + \delta)(p + q) = u(\alpha + \delta)(p + q) = 0$.

Proof We only show the necessity since the sufficiency can easily be shown. By the proof of Theorem 2.1, there exists an invertible matrix Q such that $(e_1^{'}, e_2^{'}, e_3^{'}) = (e_1, e_2, e_3)Q$ and $[e_1^{'}, e_2^{'}] = 0$, $[e_1^{'}, e_3^{'}] = e_1^{'} + \beta^{'} e_2^{'}$, $[e_2^{'}, e_3^{'}] = \delta^{'} e_2^{'}$. We use the notations before Lemma 1.2. By [11, Proposition2.4], $k_{i3}^{'} = k_{3i}^{'}$ for i =1, 2, 3, which implies that $k_{3i} = k_{i3}$ for i =1, 2, 3 by Lemma 1.2.

(I) If A is similar to $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix}$, then $\beta^{'} = 0$ and $\delta^{\prime} = 1$. By [11, Proposition 2.4], we have Part (I).

(II) Let A be not similar to $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix}$.

(a). If $z \neq 0$, then $k_{33}^{'} \neq 0$ Thus $k_{12}^{'} = k_{21}^{'}$ by [11, Proposition 2.4], which implies $k_{12} = k_{21}$. It is straightforward to check that relation (1)-(22) in the proof of Theorem 2.1 hold iff $\alpha us + \alpha pz + \gamma u^2 + \gamma yz + \beta s^2 + \beta xz + \delta su + \delta zp = 0$.

(b). If z =0, then we can obtain that r is the second case in Part (II) by using the method similar to the proof of Theorem 2.1. □

4 Coboundary Lie bialgebras

In this section, using the general solution, which are obtained in the section above, of CYBE in Lie algebra L with dim L =3 and dim L' =2, we give the sufficient and necessary conditions which ($L, [\,], \Delta_r, r$) is a coboundary (or triangular)Lie bialgebra.

Theorem 4.1 *Let L be a Lie algebra with a basis e_1, e_2, e_3 such that $[e_1, e_2]$ =0, $[e_1, e_3] = \alpha e_1 + \beta e_2$, $[e_2, e_3] = \gamma e_1 + \delta e_2$, where $\alpha, \beta, \gamma, \delta \in k$, and $\alpha\delta - \beta\gamma \neq 0$. Set $A = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}$. Let $p, u, s \in k$, $r \in Im(1-\tau)$ and $r = p(e_1 \otimes e_2) - p(e_2 \otimes e_1) + s(e_1 \otimes e_3) - s(e_3 \otimes e_1) + u(e_2 \otimes e_3) - u(e_3 \otimes e_2)$. Then*

(I) ($L, [\,], \Delta_r, r$) is a coboundary Lie bialgebra iff

$$(s, u) \begin{pmatrix} \beta\delta + \alpha\beta & -\beta\gamma - \alpha^2 \\ \delta^2 + \gamma\beta & -\delta\gamma - \gamma\alpha \end{pmatrix} \begin{pmatrix} s \\ u \end{pmatrix} = 0;$$

(II) If two characteristic roots of A are equal and A is similar to a diagonal matrix in the algebraic closure P of k with char k =2, then ($L, [\,], \Delta_r, r$) is a triangular Lie bialgebra for any $r \in Im(1-\tau)$;

(III) If the condition in Part(II) does not hold, then($L, [\,], \Delta_r, r$) is a triangular Lie bialgebra iff $-\beta s^2 + \gamma u^2 + (\alpha - \delta)us = 0$.

Proof We can complete the proof as in the proof of [10, Theorem 3.3]. □

Example 4.2 *Under Example 2.2, and $r \in Im(1-\tau)$, we have the following :*

(I) ($L, [\,], \Delta_r, r$) is a coboundary Lie bialgebra iff $r = p(e_1 \otimes e_2) - p(e_2 \otimes e_1) + s(e_1 \otimes e_3) - s(e_3 \otimes e_1) + u(e_2 \otimes e_3) - u(e_3 \otimes e_2)$ with $a(s^2 + u^2) = 0$;

(II) ($L, [\,], \Delta_r, r$)is a triangular Lie bialgebra iff $r = p(e_1 \otimes e_2) - p(e_2 \otimes e_1)$.

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